On the Stress-optical Effect in Transparent Solids strained beyond the Elastic Limit.

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(Abstract.)

Introduction.

Experiments carried out by one of the present authors with a slab of glass under flexure have indicated that artificial double refraction in glass strained beyond the elastic limit is probably proportional to the stress rather than to the strain.* More recently, the experiments of Prof. E. G. Coker and Mr. K. C. Chakko† have suggested that in celluloid or xylonite, whilst for the highest loads the double refraction is no longer proportional to either the stress or the strain, it is more nearly proportional to the former than to the latter.

Further, casual observations in the past had shown one of us that, when a piece of transparent material, whether glass or celluloid, which had been overstrained, was released, a certain amount of residual illumination was visible between crossed Nicols in certain cases. The illumination gradually died out, showing that the artificial double refraction exhibited some of the characteristics of permanent set, with a slow recovery. No precise measurements of this effect, however, seem ever to have been made.

In the few isolated cases observed by one of us the stress distribution was a complex one. In such a case the residual optical effect on removal of the load is the consequence of a mechanical readjustment of stresses and is not a true optical residual, that is, a double refraction persisting after the stress has been entirely removed.

It seemed therefore of interest to examine whether, in fact, such optical permanent set did exist, as if so it would apparently settle the question whether stress or strain is the immediate cause of double refraction.

In addition, this opened up the whole question of time effects in this connection, of which apparently no observations had been made to date, and it was hoped that a study of such effects might throw some light on the mechanism of photo-elasticity, of which very little is really known.

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* *Cf.* L. N. G. Filon, 'Phil. Trans.,' A, vol. 207, pp. 303–305.

† "The Stress-Strain Properties of Nitro-Cellulose and the Law of its Optical Behaviour," 'Phil. Trans.,' A, vol. 221, pp. 139–162.

The plan adopted was to observe a test-piece under simple stress (pressure in the case of glass and tension in that of xylonite or celluloid) between crossed Nicols. Sodium light was employed throughout and changes of relative retardation were measured by the shift of the black band produced by a Babinet compensator.

In such a case of simple loading the applied stress should disappear throughout immediately the load is removed, *provided the material is homogeneous.*

As the experiments progressed, new lines of enquiry suggested themselves, so that, ultimately, the whole subject of time-effect, or creep, for strain as well as for double refraction, came under consideration, more particularly in xylonite.

Experiments on Glass.

Rectangular blocks of optical glass of specific gravities ranging from 2·76 to 4·78 had been supplied before the War by Messrs. Zeiss, and these specimens were subjected to simple pressure between two knife-edge blocks. The loads applied approached, in most cases, as near to the breaking point as we dared go (150 to 220 kgrm.-wt. per square centimetre), and were left on for a period of 24 hours or more, the relative retardations produced ranging from three wave-lengths for the lighter glasses to one wave-length for the heaviest glass.

The apparatus employed allowed of the load being put on and removed very rapidly and smoothly, the optical effect being observed at the same time, while the retardation could be measured to an accuracy of 0·01 wave-length; but in no case was there any increase in the observed retardation during the time the load was left on, nor was there any trace of residual retardation on removal of the load.

No direct measurement was made of the strain in the glass, as the experimental difficulties in this respect have not yet been overcome, but experiments described elsewhere* indicate that plastic flow in glass sets in at much lesser stresses than those here employed.

So far, then, as these experiments go, there is no indication of any residual stress-optical effect, or that the artificial double refraction in glass at ordinary temperatures is dependent upon any other factor than the stress actually applied.

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* H. T. Jessop, 'Phil. Mag.,' vol. 42, October, 1921.

Observations on Xylonite.

We next examined xylonite, where residual effects previously observed had been much more marked. In this case the specimens were observed under tension, as pressure could not be applied without causing buckling.

An extensometer was devised which permitted of the observation of simultaneous optical effect and strain, and we found a considerable time-effect in both, under stresses ranging from 67·6 to 236·4 bars.

The specimen was loaded at a given instant, as quickly and smoothly as possible, and timed observations of the stress-optical effect and strain were taken over a given period (generally one hour, though in some cases the observations were continued for several days, after which time the creep was found to have practically ceased). At the end of this period, which we call the "loading" period, and during which the load was kept constant, the load was removed, and timed readings taken during the "unloading" or "recovery" period.

On loading, the observations of both stress-optical effect and strain showed in every case an immediate initial effect, roughly proportional to the load applied, followed by a marked creep. This creep was at first so rapid that no actual reading of the initial effect could be taken, and the value of this initial effect could consequently only be deduced from a graph.

On unloading, a similar immediate effect, followed by creep, was observed. The initial recovery was roughly equal to the initial effect on loading, but in general the rate of creep on recovery was not the same as that on loading. If the load is taken off after many hours, so that the specimen has had time to settle down, both strain and retardation on recovery give a curve identical with the corresponding curve on loading reversed. But if the load is taken off at an earlier stage, both strain and retardation on recovery follow a different law, which depends upon the time during which the original load was maintained. The form of this law has not yet been determined by us.

The optical creep on loading is comparatively large and may easily amount to 10 per cent. of the initial value in a few minutes. This percentage, however, diminishes as the load is decreased. These facts have an important bearing on stress determinations from observations of artificial double refraction in celluloid.

Curves of strain and relative retardation were then plotted to time, and attempts were first made to fit an exponential formula of the type

$$r = \alpha - \beta e^{-kt},$$

this being the law that would be expected, assuming the stresses in the material to be composed of elastic and viscous parts, obeying Hooke's and Stokes' laws respectively.

A thorough examination of a number of curves, however, showed large and systematic divergences from this law, and it was definitely discarded.

Eventually, after trying many formulæ, including combinations of exponential, logarithm, and power, all of which proved unsuitable for various reasons, we found that the loading curves were very well fitted by formulæ of the type

$$s = s_0 + at^{1/3} + bt \tag{1}$$

for the strain, and

$$r = r_0 + pt^{1/3} + qt \tag{2}$$

for the stress-optical effect.

When the retardation was plotted to simultaneous strain, it was found to give quite a good straight line, leading to the relation

$$r = \alpha \mathrm{T} + \beta s \tag{3}$$

where T is the stress applied to the specimen, α and β being either constant for a given specimen at a given period of its history or, at any rate, slowly varying functions of T.

This law was also found to hold for the values of r and s on unloading, T then being zero in every case.

The values of the constants in formulæ (1), (2), and (3) were found to vary from specimen to specimen, and also to be affected by previous treatment of the specimen and by lapse of time, the variation being such as to indicate a progressive change with time in the physical properties of the xylonite.

On repeating the experiments after about six months, the constants a and p, which measure the bulk of the creep, at any rate in the earlier stages, were found to be practically halved. Thus the importance of using old and well-seasoned specimens in stress investigations cannot be too much emphasised.

The law (3) is satisfactorily explained on the assumption that xylonite is made up of two constituents, one perfectly elastic and the other plastic, the stress-optical effect in each being strictly proportional to the applied stress, but the two components having different stress-optical coefficients.

In order to test whether this heterogeneity were uniformly distributed throughout the material, or whether the two components were provided by an elastic skin enclosing a plastic core, two similar specimens were tested, one specimen having a thickness of 1 mm. removed from each face immediately before loading.

The differences found between the constants α and β for the two specimens were less than many which we had found between different sets of observations on the same specimen.

It would appear, then, that the "skin" and "core" could not produce the effect observed, and that if the two-material hypothesis is correct, then xylonite consists of an intimate mixture of two constituents, with different elastic and plastic properties and different stress-optical coefficients. It is further possible that the two materials are allotropic modifications of the same substance, and that their proportions are altered by the action of stress and other causes. This might account for the observed divergences in the values of α and β under different conditions.

The Rotation of Two Circular Cylinders in a Viscous Fluid.

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In a previous communication* we employed the solution of the equation $\nabla^4\psi = 0$ in bipolar co-ordinates defined by

$$\alpha + i\beta = \log \frac{x+i(y+a)}{x+i(y-a)} \tag{1}$$

to discuss the problem of the elastic equilibrium of a plate bounded by any two non-concentric circles.

There is a well-known analogy between plain elastic stress and two-dimensional steady motion of a viscous fluid, for which the stream-function satisfies $\nabla^4\psi = 0$. The boundary conditions are, however, different in the two cases, and the hydrodynamical problem has its own special difficulties.

In this paper we discuss the problem of the rotation of two parallel infinite circular cylinders in a viscous fluid. The solution is different in form according as one cylinder does, or does not, enclose the other. In the former case the problem can be solved in finite terms; while in the latter the problem is in general insoluble; that is to say, except in special circumstances, there is no *steady* motion which satisfies all the necessary conditions.

Let the two cylinders be defined by constant values of α, say, $\alpha = \alpha_1, \alpha_2$. We may take α_1 positive and greater than α_2, then α_2 will be positive or negative according as the second cylinder does, or does not, enclose the first. Let the cylinders rotate about their axes with angular velocities ω_1, ω_2. The stream-function ψ satisfies $\nabla^4\psi = 0$ when steady motion is established, and

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* "Plane Stress and Plane Strain in Bipolar Co-ordinates," 'Phil. Trans.,' A, vol. 221, p. 265 (1920). Reference to this paper will be denoted by T.